14.1.2 Pseudo-arc

A. Lelek, ZBIORY (Sets), PZWS Warszawa 1966, pages 90-95.The first heredi-

tarily indecomposable continuum has been constructed by B. Knaster in 1922. This continuum K, discovered by Knaster, is just named the *pseudo-arc* or *K-naster's continuum*. The pseudo-arc K is obtained as the common part of a decreasing sequence of continua in the plane, every of which is the union of finitely many discs, forming a chain of sets. We will consider chains

$$\mathcal{D} = (D_1, D_2, \ldots, D_k)$$

of discs D_i in the plane, having the property that any two consecutive disc D_i and D_{i+1} have interior points in common, i.e., that their intersection — nonempty by assumption — is not contained in the boundary of any of them (Figure A). Discs D_i are called *links* of the chain $\mathcal{D}$. The union of all links of the chain $\mathcal{D}$ is a continuum. Observe that if we delete from the chain $\mathcal{D}$ a link which is not an end link, i.e., a link D_i such that $1 < i < k$, then the union of the remaining links is a not connected set, having two components.

Let two chains of discs $\mathcal{D}$ and $\mathcal{D}'$ be given. We say that the chain $\mathcal{D}'$ *refines* the chain $\mathcal{D}$ if each link D'_s of the chain $\mathcal{D}'$ is contained in the interior of at least one link D_i of the chain $\mathcal{D}$, i.e., if for every $D'_s \in \mathcal{D}'$ there exists a link $D_i \in \mathcal{D}$ such that $D'_s \subset D_i$ and that the disc D'_s has no common points with the boundary of the disc D_i (Figure B).

We say that the chain $\mathcal{D}'$ is *crooked* in the chain $\mathcal{D}$ if it refines $\mathcal{D}$ and if, for every pair of links D_i and D_j of the chain $\mathcal{D}$ such that

$$i + 2 < j, \tag{1}$$

and for every pair of links D'_s and D'_v of the chain $\mathcal{D}'$, intersecting the links D_i and D_j respectively, i.e.,

$$D_i \cap D'_s \neq \emptyset \neq D_j \cap D'_v,$$

there are, in the chain $\mathcal{D}'$, links D'_t and D'_u, lying between the link D'_s and D'_v in the same order, i.e.,

$$s < t < u < v \quad \text{or} \quad s > t > u > v,$$

such that

$$D'_t \subset D_{j-1} \quad \text{and} \quad D'_u \subset D_{i+1}. \tag{2}$$

In other words, a part of the chain $\mathcal{D}'$ which is contained between the links D'_s and D'_v has to form a fold in the chain $\mathcal{D}$ between the links D_i and D_j (Figure C). The chain $\mathcal{D}'$, to go from the link D_i to the link D_j has first to come to the link D_{j-1}, next go back to the link D_{i+1}, and only after this it can reach the link D_j. Such a fold must exist for every two links D_i and D_j having no adjoining (i.e. neighbor) link in common. Now, let there be given in the plane